

THE PRACTICE OF DELTA-GAMMA VaR: IMPLEMENTING THE QUADRATIC PORTFOLIO MODEL[†]

GIUSEPPE CASTELLACCI AND MICHAEL J. SICLARI

ABSTRACT. This paper intends to critically evaluate state-of-the-art methodologies for calculating the VaR of non-linear portfolios from the point of view of computational accuracy and efficiency. We focus on the quadratic portfolio model, also known as “Delta-Gamma,” and, as a working assumption, we model risk factor returns as multi-normal random variables. We present the main approaches to Delta-Gamma VaR weighing their merits and accuracy from an implementation-oriented standpoint. One of our main conclusions is that the Delta-Gamma normal VaR may be less accurate than even Delta VaR. On the other hand, we show that methods that essentially take into account the non-linearity (hence gammas and third or higher moments) of the portfolio values may present significant advantages over full Monte Carlo revaluations. The role of non-diagonal terms in the Gamma matrix as well as the sensitivity to correlation is considered both for accuracy and computational effort. We also qualitatively examine the robustness of Delta-Gamma methodologies by considering a highly non-quadratic portfolio value function.

KEYWORDS AND PHRASES: *Value at risk, Risk analysis, Risk management, Finance, Stochastic processes, Simulation, Delta-Gamma-Theta VaR, Quadratic portfolios.*

JEL CLASSIFICATION: *C10*

1. INTRODUCTION

Value-at-Risk, or *VaR* has rapidly become the standard quantitative benchmark for measuring the risk exposures of financial portfolios. Such trend has been institutionalized and rather standardized.¹ VaR is usually defined as the loss in the value (in a given currency) of a portfolio that would result

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¹The work and recommendations of the Basel Committee on Banking Supervision under the auspices of the Bank for International Settlements are the most important evidence of institutionalization. For one of the most recent accounts, the reader may look up [14]. As for methodological standardization, RiskMetrics is emblematic (cf. [16]).

from an adverse movement in the relevant market factors of its components within a given time horizon and confidence level. As such, VaR can be considered within the wider perspective of coherent risk measures (cf. [1]). Despite several generalizations and theoretical developments many issues remain open when one attempts to implement VaR calculators for realistic portfolios. Most of the problems originate both from the non-linearity of the constituent instruments as well as from the computational effort deriving from the number size and complexity of valuation formulas involved.

The aim of this paper is to comparatively analyze issues that arise in the software implementation of VaR calculators based on the most common incarnations the quadratic portfolio model.² Assuming that portfolio values are quadratic seems satisfactory enough, for the non-linearity of most derivative contracts is well approximated quadratically and such approximations aggregate over a portfolio.

In Section 2 we briefly define VaR as an appropriate quantile of the portfolio value changes distribution. Because we are concerned with the implementation of Delta-Gamma VaR methodologies rather than with the modeling of the risk factors per se, we stick with the widespread assumption that risk factor returns are multi-normal. In the following section, we introduce five methodologies that stem from the quadratic portfolio assumption and that we are going to test.

In section 4 we begin by observing how the computational burden due to the presence of cross-gamma terms can be reduced with suitable techniques. The computational issue of cross gamma terms has not been raised to the best of our knowledge, and all the numerical examples we have encountered in the literature assume that the gamma matrix is diagonal.

²These methodologies are often termed by prefixing the appellative *Delta-Gamma* to some specific qualifier. *Delta-Gamma Normal* VaR, for example, denotes the methodology that assumes the portfolio values are quadratic and normally distributed (an assumption obviously incompatible with the normality of risk factor return, yet commonly made). Essentially, we will use the terminology "quadratic portfolio model" and "Delta-Gamma" interchangeably.

Section 5 applies the five³ VaR methodologies considered earlier to five different portfolios. Each portfolio was tailored to capture a specific attribute of downside risk. The accuracy of each method is evaluated across the entire correlation range. Two interesting conclusions emerge. Contrary to a somewhat widespread belief among practitioners, Delta-Gamma Normal VaR may yield results that are less accurate than even Delta (linear) VaR. Also, by allowing a suitable (higher) number of risk factors, Delta-Gamma methodologies can accurately tackle portfolios that seem highly non-quadratic.

Section 6 goes on to demonstrate the computational requirements of each VaR methodology as a function of the number of risk factors as well as the number of instruments (transactions) within the portfolio. The final Section 7 contains a few conclusive remarks.

Several authors have tackled the problem of calculating VaR within the quadratic portfolio model. Britten-Jones and Schaefer [2] seem to have first applied the theory of quadratic forms in normal variables to estimate higher moments of a quadratic portfolio. Mina and Ulmer [13] review the main methodologies and favor taking the inverse (fast) Fourier transform of the characteristic function of the portfolio value function. More recently, Duffie and Pan [5] greatly extend this approach to cover returns with jumps and defaults. Jaschke [8] focuses on the accuracy and computational effort of Delta-Gamma Cornish-Fisher. His conclusions seemingly agree with ours insofar as the usage of the Cornish-Fisher expansion is concerned.

We believe, that the work here presented has a similar goal to Pritsker's work [15].⁴ The parametric methodologies considered in [15] seem not to

³A finer classification would count 8 different methodologies, for Delta-Gamma Monte Carlo, Delta-Gamma-Normal, and (Delta-Gamma-)Cornish-Fisher have each a diagonal and full version depending on whether one assumes the Gamma matrix to be diagonal.

⁴An earlier version of the paper incorporated material from [3]: a survey of quadratic portfolio VaR methodologies, several analytic closed form VaR formulas, and a two-factor analytic case study that illustrated how some of the empirical results may be also proven mathematically. This extended the original scope of the paper toward a direction already

use the potentialities of the quadratic portfolio model at best. In particular, the so called Delta-Gamma-Delta methodology is a particular case of Delta-Gamma-Normal with the assumption that the risk factor changes are uncorrelated and normally distributed. The Delta-Gamma-Minimization minimizes quadratic portfolio values subject to a spherical constraint that comes from a χ_n -squared distribution. The author claims that “there is little reason to believe that this distributional assumption will be satisfied.” We cover the same Monte Carlo simulation methods as in [15] with the exception of *grid Monte Carlo*. The author proceeds to test the different methodologies on a portfolio of long calls and a portfolio of short calls. (For which there is an analytic formula in the European case, cf. [3].) Our evaluation of the methods is quite different in nature and aim. While the author provides several statistical metrics we compare VaR computation accuracy and efficiency with graphs that reflect behavior with respect to correlation, cross-gamma effects, and computational time.⁵

Overall, we hope that our endeavor to clarify and analyze the VaR computation process for nonlinear portfolios from the definition to the implementation issues will benefit the researcher as well as the practitioner.

2. DEFINING VAR

In this section we recall the basic definitions and assumption that underlie our approaches. We consider portfolios whose value depend on n risk factors X_1, X_2, \dots, X_n . We assume that the portfolio value is completely

taken in [2] and [13]. After several referee suggestions, we opted for focusing on the empirical result here.

⁵As far as accuracy is concerned, we took our full Monte Carlo simulation as our “true” value of the VaR (population quantile). A statistical measure of the accuracy of Monte Carlo estimation itself was provided through an estimate of the associated standard error (cf. Section 5). At the behest of one of the referees, we also added a symmetric confidence interval around our Monte Carlo estimates with semi-length equal to the previous standard error sample estimate.

determined by a twice differentiable function of n variables⁶, the *portfolio value function*

$$Y = PV(X_1, X_2, \dots, X_n). \quad (2.1)$$

The problem of determining VaR can be subdivided in two main stages. Modeling the input, the risk factors, and, given the input, modeling the distribution of the output, that is the portfolio value PV .⁷ Therefore, the (dependent) random variable Y admits some heretofore unknown distribution, which motivates the rather standard definition of VaR as follows.⁸

Definition 2.1. For a portfolio with portfolio value function $Y = PV(\mathbf{X})$, with cumulative distribution function F , VaR at a confidence level c is the opposite in sign of the value of the $(1 - c)$ -quantile:

$$VaR_c = -F^{(-1)}(1 - c), \quad (2.2)$$

where $F^{(-1)}$ is the quasi-inverse of the cumulative distribution function.⁹

Since we are concerned with the validity, accuracy, and efficiency of a model that attempts to capture the non-linearity of the portfolio value function, we do not focus on the choice of an appropriate dynamics for the risk

⁶Generally, the variables depend on time. Technically they can be thought of as the components of a random vector sampled from random process $\{\mathbf{X}(t)\}_{t \geq t_0}$ where $\mathbf{X}(t) = [X_1(t), \dots, X_n(t)]^\top$.

⁷For a more detailed discussion of this schematization of the VaR calculation process and the assumptions that pertain to the modeling of both risk factors and portfolio value function, cf. [3].

⁸Our choice to define VaR as the negative of a quantile is quite standard among practitioners and theorists ([1]). We encountered other definitions of VaR such as the absolute value of a quantile ([4]) or a complementary quantile ([15]), which coincides with the quasi-inverse of the survival function $S = 1 - F$ at $1 - c$.

⁹Recall that this is defined by requiring that $F(F^{(-1)}(c)) = c$ and

$$F^{(-1)}(c) = \inf\{x \in [-\infty, \infty] : F(x) \geq c\} = \sup\{x \in [-\infty, \infty] : F(x) \leq c\}.$$

factors. Therefore, we stick with the widespread assumption that the risk factor returns are multi-normally distributed.¹⁰ More formally,

Assumption 1. *We assume that the risk factors returns*

$$R_i = \Delta X_i / X_i = (X_i(T) - X_i(T_0)) / X_i(T_0) \quad (\text{for } i = 1, \dots, n) \quad (2.3)$$

are multi-normally distributed with zero mean, conditionally upon the information available at the present time T_0 that is,

$$\mathbf{R} | \mathcal{J}_{T_0} \sim N_n(\mathbf{0}, \Sigma). \quad (2.4)$$

where \mathcal{J}_{T_0} is the σ -field of information available up to time T_0 , Σ is the covariance matrix,¹¹ and $T - T_0 > 0$ is the time horizon.

Remark 2.1 (Consequences for analytic and simulation-based computation).

This assumption has direct consequences for any methodology that computes VaR analytically, for it affects the distribution of the portfolio values. On the other hand, simulation methods—and Delta-Gamma Monte Carlo VaR, in particular—are essentially independent of the specific a distributional assumption for the risk factors. Indeed, given a mechanism for generating (e.g., pseudo-random number generators) jointly distributed risk factors, one can easily apply a simulation scheme to generate samples of portfolio values and estimate VaR.¹²

¹⁰For short time horizons, this is approximately consistent with a n -dimensional Geometric Brownian Motion dynamics (with possibly time-changing parameters) for risk factors.

¹¹Our assumption can be seen as a consequence of the stipulation that the returns on the market factors form an n -dimensional Brownian motion, with corresponding Brownian filtration $\{\mathcal{J}_T\}_{T \geq 0}$. We also abuse vector notation and write $\mathbf{R} = \Delta \mathbf{X} / \mathbf{X}$.

¹²One of the reason for the choice of normal or log-normal risk factors—which is also our working assumption—is that multi-normal pseudo-random number generators are

3. DELTA-GAMMA-(THETA) VAR

The so called *Delta-Gamma-(Theta) VaR* is the collection of VaR computation methodologies that stem from the quadratic portfolio model. This modelling assumption does not require normality of the risk factors¹³

Assumption 2 (Quadratic Portfolio). *Henceforth we assume the portfolio value function is such that*

$$\Delta PV = \frac{1}{2} \sum_{i,j=1}^n \Gamma_{ij} \Delta X_i \Delta X_j + \sum_{i=1}^n \delta_i \Delta X_i + \theta \Delta t. \quad (3.1)$$

One sets

$$\delta_i = \frac{\partial PV}{\partial x_i}$$

$$\Gamma_{ij} = \frac{\partial^2 PV}{\partial x_i \partial x_j}.$$

readily available. Several other distributions have been attempted, especially leptokurtic ones. For example, [11] consider α -stable distributions to this problem. Stochastic Simulation methods only require jointly distributed random numbers. However, analytic approximation methods require some distributional information on the portfolio values. In this setting, generally little is known for nonlinear portfolios, and most authors confine their studies to linear portfolios. In particular, below we mention the characterization of distribution of quadratic forms in non-normal variables as an open problem. Progress in this direction would allow analytic approximation of (quadratic portfolio) VaR with non-normal factors.

¹³Notice that this modelling assumption does not require normality of the risk factors. This seems to contradict a claim in [15] to the effect that all parametric Delta-Gamma methodologies are doomed to inaccuracy in view of the normality of risk factors assumption. Insofar as simulation is concerned, the independence of this assumption of the risk factor distribution is already apparent in Delta-Gamma Monte Carlo. The development of analytic methods that apply the quadratic portfolio model to non-normal risk factor is still an open problem (cf. [3]).

and $\theta = \partial PV / \partial t$, so that this model identifies ΔPV with its second order Taylor polynomial in the risk factors approximation.¹⁴ Using the *return adjusted* deltas and gammas $\tilde{\delta}_i = \delta_i X_i$ and $\tilde{\Gamma}_{ij} = \Gamma_{ij} X_i X_j$, we can write this expression in terms of the returns $R_i = \Delta X_i / X_i$:

$$\Delta PV = \frac{1}{2} \sum_{i,j=1}^n \tilde{\Gamma}_{ij} R_i R_j + \sum_{i=1}^n \tilde{\delta}_i R_i + \theta \Delta t. \quad (3.2)$$

Notice that this equation can be rewritten more succinctly in matrix form:

$$\Delta PV = \frac{1}{2} \mathbf{R}^\top \tilde{\mathbf{\Gamma}} \mathbf{R} + \tilde{\boldsymbol{\delta}}^\top \mathbf{R} + \theta \Delta t, \quad (3.3)$$

where $\tilde{\mathbf{\Gamma}} = [\tilde{\Gamma}_{ij}]$.

Below we proceed to illustrate the most common approaches to computing VaR within the quadratic portfolio model, which we are later going to comparatively test computationally.

3.1. Delta-Gamma Monte Carlo Simulation. The quadratic portfolio assumption immediately leads to a version of Monte Carlo simulation, i.e. *Delta-Gamma Monte Carlo*¹⁵ simulation, that is conceptually simple, significantly more accurate than most analytic methodologies, and less computationally expensive than full Monte Carlo.¹⁶ This is readily obtained by observing that (3.3) gives an explicit expression of ΔPV in terms of the risk factor returns. One can therefore plug in (3.3) \mathbf{R} 's values drawn from a fixed distribution with correlation structure given by the covariance matrix Σ .

¹⁴Ignoring higher order terms depending on time is consistent with the assumption that the risk factor returns approximately follow a Brownian motion—an assumption that we make but is not necessary.

¹⁵This is also called *Partial Monte Carlo* (cf., e.g., [13]).

¹⁶By *Full Monte Carlo* we denote the methodology that involves randomly drawing risk factors from a given distribution or historical sample or other scheme, feeding the values obtained for every draw to the corresponding valuation formulas or algorithms, computing the resulting portfolio values and finally gleaning the sought for percentile from the empirical distribution of such portfolio values.

Remark 3.1 (Advantages of Delta-Gamma Monte Carlo). It is important to highlight the main features of this methodology as opposed to other parametric and simulation methods. To this effect, note that Delta-Gamma Monte Carlo

- does not require distributional assumptions on the risk factors. In particular, risk factors can be drawn from skewed and leptokurtic distributions or historical samples.
- does not require the computation of the moments of the distribution of ΔPV (unlike most parametric methodologies).
- does not require exact revaluation of every position in a given portfolio (unlike full Monte Carlo).

From the point of view of balancing accuracy versus computational efficiency, this methodology should lie between full Monte Carlo and the main parametric Delta-Gamma methodologies. We expect, therefore, this methodology to be more accurate than most parametric Delta-Gamma methodologies, but less accurate than full Monte Carlo. On the other hand, this methodology should be more computationally expensive than most parametric methodologies, but less expensive than full Monte Carlo. Our expectations are corroborated by the numerical validations and analysis of Section 5 through 7.

3.2. The Moments of ΔPV . Most analytic methods for computing Delta-Gamma VaR as well as some non-parametric methods require calculation of the moment of a distribution. Here we briefly recall expressions for the first three moments of ΔPV . These can be seen as elementary applications of the theory of quadratic forms in multi-normal random variables.¹⁷

¹⁷This is by now classical. For more details, cf., e.g., [9] and [12]. [2] seems to have first applied those results to Delta-Gamma VaR.

The mean can be readily computed from (3.3):

$$\mu_{PV} = \frac{1}{2} \text{trace}(\tilde{\Gamma}\Sigma) + \theta\Delta t. \quad (3.4)$$

The variance of ΔPV can be calculated to be

$$\sigma_{PV}^2 = \mu_2(\Delta PV) = \frac{1}{2} \text{trace}((\tilde{\Gamma}\Sigma)^2) + \tilde{\delta}'\Sigma\tilde{\delta}. \quad (3.5)$$

For the third moment we obtain

$$\mu_3(\Delta PV) = \text{trace}(\tilde{\Gamma}\Sigma)^3 + 3\tilde{\delta}'\Sigma\tilde{\Gamma}\Sigma\tilde{\delta}. \quad (3.6)$$

3.3. Delta-Gamma Normal VaR. Perhaps, the most widespread analytic Delta-Gamma methodology is moment matching of the distribution of a quadratic ΔPV within the 2-parameter normal family $N(\mu_{PV}, \sigma_{PV}^2)$, also known as *Delta-Gamma-Normal VaR*. Once the moments are estimated, VaR is calculated in a fashion similar to Delta (linear) VaR:

$$VaR_c = -(\mu_{PV} + \sigma_{PV}N_c).$$

Below, we will see numerical evidence that this method may well be less accurate than even linear methods. This shortcoming can also be proved theoretically, cf. [3].

3.4. The Cornish-Fisher Expansion. The Cornish-Fisher expansion avoids the problem of estimating percentiles of a fitted (moment matched) distribution from the sample moments, which is in general quite complicated. For a distribution G of a standardized random variable, this expansion gives an expression of $x_c = G^{-1}(c)$ in terms of $z_c = N^{-1}(c)$ (and polynomial). In particular

$$G^{-1}(c) = z_c + \frac{1}{6}(z_c^2 - 1)\kappa_3 + \frac{1}{24}(z_c^2 - 3z_c)\kappa_4 + \dots, \quad (3.7)$$

where κ_r is the r -th cumulant of the random variable. When G is the distribution of $(\Delta PV - \mu_{PV})/\sigma_{PV}$ we obtain the “standardized VaR” as

$$VaR_c^{std} = N^{-1}(1 - c) + \frac{1}{6}(N^{-1}(1 - c)^2 - 1) \text{Skewness}(\Delta PV),$$

so that the actual VaR is

$$VaR_c = |\mu_{PV} + \sigma_{PV}VaR_c^{std}|. \quad (3.8)$$

4. SOME LINEAR ALGEBRA SIMPLIFICATIONS

In the formula for the r -th moment of ΔPV there appear r -fold products of matrices (cf., e.g., [2], [9], and [3]). For a realistic portfolio (depending on a number of risk factors of the order of 10^3) calculating these products may present such a computational burden on a computing system that Monte Carlo simulation may seem a viable alternative. In fact, the matrix products in the r -th moment of ΔPV with n factors involve calculating a number of products of order $O(r, n) = rn^3$. However, in the case of the moments of ΔPV , not only the matrices involved, but also their products are symmetric. Therefore, one can reduce the computation to the upper triangular part of the matrix, a calculation of products of order $O(r, n) = rn^2(n + 1)/2$. As for the argument of the trace, we remark that if we diagonalize both $\tilde{\Gamma}$ and Σ as $\mathbf{P}_\Gamma \Lambda_\Gamma \mathbf{P}'_\Gamma = \tilde{\Gamma}$ and $\mathbf{P}_\Sigma \Lambda_\Sigma \mathbf{P}'_\Sigma = \Sigma$, then

$$\text{trace}(\tilde{\Gamma}\Sigma)^r = \text{trace}(\mathbf{P}_\Sigma \mathbf{P}_\Gamma \tilde{\Gamma} \Sigma \mathbf{P}'_\Gamma \mathbf{P}'_\Sigma)^r = \sum_{i=1}^n \lambda_{\Gamma,i}^r \lambda_{\Sigma,i}^r.$$

Therefore, if one knows the eigenvalues of $\tilde{\Gamma}$ and of Σ , one can bypass the matrix product and express the trace as a sum of product of eigenvalues. It is common practice to assume that $\tilde{\Gamma}$ is diagonal. In this case, one has only to compute the eigenvalues of Σ . We note that often VaR calculators perform principal component analysis of Σ , thereby determining the eigenvalues of this matrix. In this case the trace is essentially already computed.

The assumption that $\tilde{\Gamma}$ is diagonal is quite a delicate one. On one hand, while we expect the general $\tilde{\Gamma}$ to be block diagonal, for portfolios values can be expressed as linear combinations of q *deal* values:¹⁸

¹⁸Generally the term *deal* correspond to a derivative contract. In this decomposition it is desirable to assume further that the deals allow to express a portfolio as a linear combination of value function that are either linear in one factor or purely nonlinear (i.e., their first order Taylor polynomial is zero). One can also absorb constant terms into a single summand.

$$PV(X_1, X_2, \dots, X_n) = \sum_{k=1}^q a_k V_k(X_{i_1}, X_{i_2}, \dots, X_{i_k}),$$

where $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$ is the subset of factors on which the k -th deal depends. For most deals i_k , the number of variables on which the k -th deal depends, is rather small. The gamma matrix is obviously the direct sum of the Hessians of the individual deal values.

On the other hand, one can still develop *smart* computation methods for the individual terms of $\tilde{\Gamma}$. This avoids calculating the cross-gammas based on preliminary considerations on the structure of the portfolio that lead to the block diagonal matrix structure.

5. VALIDATION

The following five methodologies can be utilized to compute VaR:

- (1) Delta-Normal
- (2) Delta-Gamma-Normal or Delta-Gamma
- (3) Cornish-Fisher
- (4) Delta-Gamma Monte Carlo
- (5) Full Monte Carlo Simulation

Each methodology involves a different level of accuracy and computational effort. The first three methods utilize only matrix algebra while the remaining two involve simulation. The accuracy and computational speed of the first four VaR methodologies will be assessed in comparison to full Monte Carlo simulation which is assumed to be the most accurate methodology for nonlinear portfolios. All full Monte Carlo simulations were carried out using 100,000 trials, yielding an error smaller than 0.1% based on numerical comparisons with 1,000,000 trial Monte Carlo simulations. The latter, in their turn, had a sample standard error of less than 0.1%.¹⁹

Four basic test cases are utilized for this purpose and are listed below in Table 1. An additional case (a portfolio consisting of a long straddle and

¹⁹Standard errors were estimated using the same method followed by [4, p.165], that is taking a sample standard deviation of a sample of full Monte Carlo simulations.

Case	VaR Scenarios	No. of Options
1	Long/Short ATM European Calls/Put	1
2	ATM Straddle Long Call and Put	2
3	ATM Calls or Puts	2
4	ATM Spread Call Option	1
5	Long Straddle & Short Strangle	4

TABLE 1: Basic Test Cases

a short strangle) is considered to illustrate the robustness of the quadratic portfolio model with respect to violations of the quadratic nature of portfolio values. All options except the spread option are evaluated as Black-Scholes European options.

Since our aim is to test quadratic portfolio VaR methodologies rather than the modeling of risk factors per se, we are using parameter values that are realistic but synthetic.²⁰ In all of our market scenarios the short term risk-free rate as well as the convenience yield rate are assumed to be zero. Table 2 lists the other parameter values used for our portfolio positions in the first three examples.²¹ These correspond to the input of the Black-Scholes formula. In particular, σ_{OPT} denotes the annualized volatility used in the option pricing, whereas σ_{VAR} stands for the daily volatility that was employed in the risk factor simulations. Finally, all the VaR are computed for a one-day time horizon.

Distribution-free estimation of confidence intervals based on order statistics (cf. [10, Section 20.38]) were also attempted, but the size of our sample rendered the procedure computationally infeasible.

²⁰In particular, such is the covariance matrix Σ , as represented by the volatilities and correlations of risk factors.

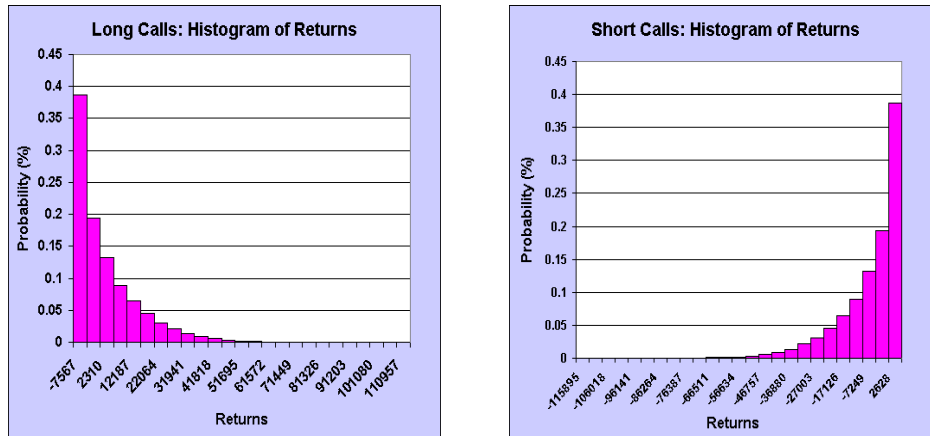
²¹The last two test cases need more specific parameters and input values. We will digress on those in the appropriate subsections.

Parameter	Symbol	Value (Unit)
Time to Expiry	T	0.1 (yr.)
Initial Price	P	100 (\$)
Strike Price	X	100 (\$)
Volatility	σ_{OPT}	0.3
VaR volatility	σ_{VAR}	0.1
Notional	Not	1000

TABLE 2: List of the Main Parameter Values for the First Three Test Cases

5.1. Test Case 1: Long vs. Short Call or Put. Figure 1 shows histograms of returns for an ATM long call and a short call that were computed using full Monte Carlo simulation. The histograms are highly skewed.

Figure 2 shows a comparison of the computed VaR for long vs short calls



(A) Long Calls

(B) Short Calls

FIGURE 1: Returns Histogram for Long and Short Calls.

and long vs short puts using all five methods. Delta-Normal cannot distinguish between a long or a short call or put yielding the same VaR for either situation. Once Gamma is introduced into the method, the method now distinguishes between a long or a short call or put. The VaR is greater for a short call in comparison to a long call. This reflects the fact that buying a

call (i.e. long a call) has a limited risk on the downside while selling a call (i.e. short a call) has a higher unbounded upside risk. A similar situation exists for puts. Cornish-Fisher and Delta-Gamma Monte Carlo both slightly overpredict the VaR. Delta-Gamma Monte Carlo is in good agreement with the full Monte Carlo simulation although it slightly underpredicts risk on the long side.

5.2. Test Case 2: ATM Straddle. In this test case, the portfolio consists of an ATM call and put with two different underlying assets. Figure 3 shows two returns histograms for different correlations between the underlying risk factor returns of the call and put as computed with full Monte Carlo simulation. The histograms are skewed with the skewness depending on the level of the correlation. As the underlying assets of the call and put become highly correlated the risk or VaR should greatly diminish and become negligible with full correlation.

Figure 4 shows the skew of the returns provided by three of the nonlinear methods. The analytic formula for the skewness (of a quadratic form in random variables) used in Cornish-Fisher VaR and the sample skewness coming from Delta-Gamma Monte Carlo simulations are in very good agreement, for both VaR methods stem from the quadratic portfolio assumption. On the other hand, full Monte Carlo simulation gives somewhat lower values. These are more accurate since full Monte Carlo implicitly takes into account (all) higher order terms in the Taylor series approximation.

Figure 5 shows a comparison of the VaR computed by all five methods as a function of the correlation of the underlying assets. The highest VaR is computed when the assets are anti-correlated or have a correlation coefficient of -1 . The Delta-Normal and Delta-Gamma VaR is much higher in comparison to the remaining three methods. The Cornish-Fisher method slightly overpredicts the VaR while the Delta-Gamma Monte Carlo method slightly underpredicts the VaR. As the underlying assets become correlated,

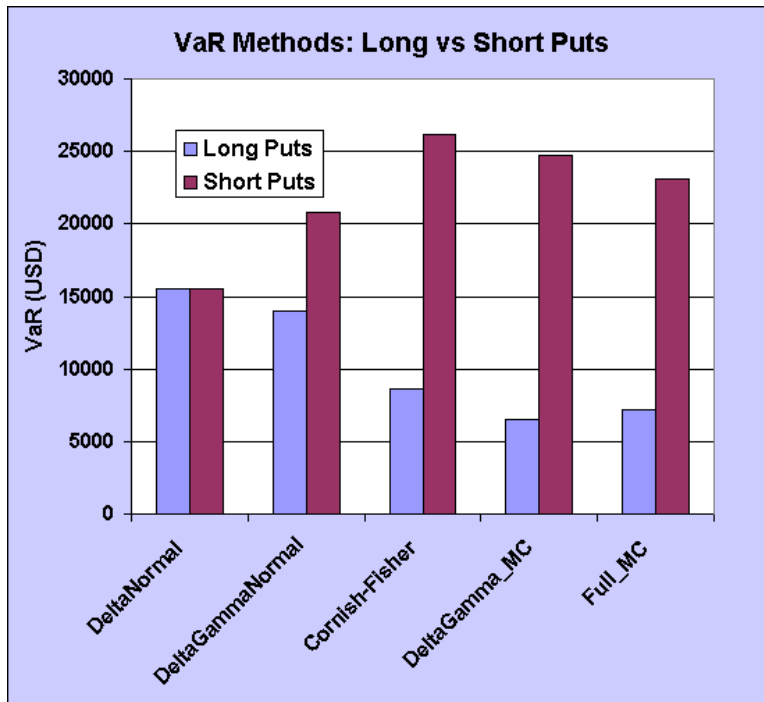
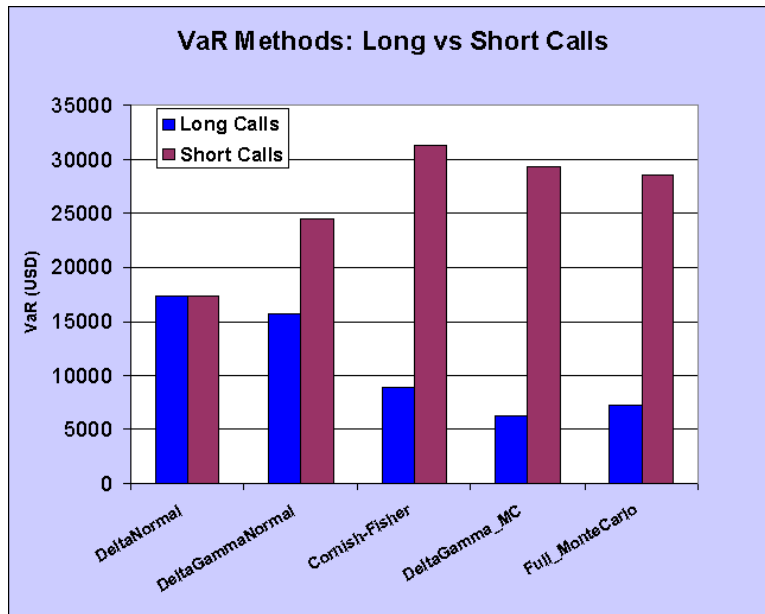


FIGURE 2: VaR Comparison for Test Case 1: Long vs Short Calls or Puts.

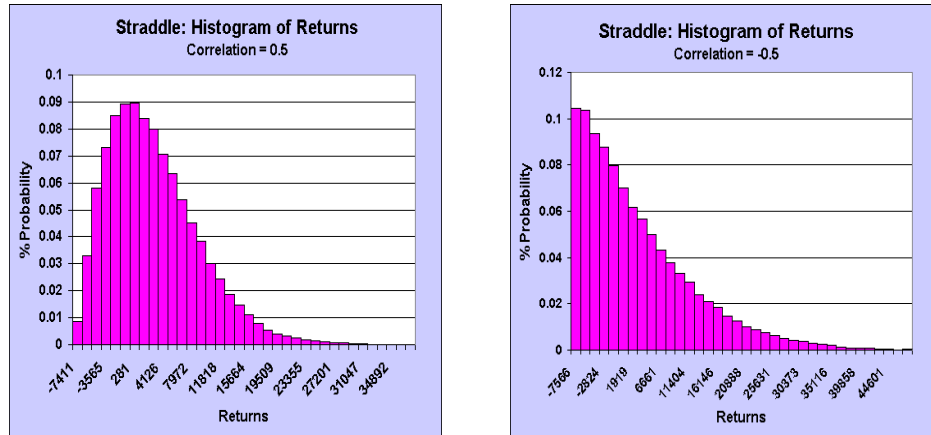


FIGURE 3: Histograms of Returns for Test Case 2: ATM Straddle with Correlation ± 0.5 .

the VaR diminishes for all of the methods except the Delta-Gamma method. The Delta-Gamma method exhibits a strange and "incorrect" behavior as the underlying assets become perfectly correlated. For perfect correlation, even the simplest linear Delta-Normal method yields a negligible VaR.

5.3. Test Case 3: Calls and Puts. In this test case, we consider two portfolios, each consisting of two ATM calls or two ATM puts with two underlying and correlated risk factors. Figure 6 shows the histogram of returns computed by full Monte Carlo simulation for anti-correlated calls (i.e. correlation coefficient = -1) and uncorrelated calls (i.e. correlation coefficient = 0). The histograms of returns are highly skewed with the degree of skewness largely dependent upon the correlation of the underlying assets.

Figure 7 shows a comparison of the VaR computed by all five methods for a portfolio of two calls and a portfolio of two puts as a function of the correlation of the underlying assets. In this test case, the two calls or two puts act like a straddle when they are anti-correlated. Hence, the VaR diminishes and becomes negligible as the correlation coefficient approaches -1 . In this

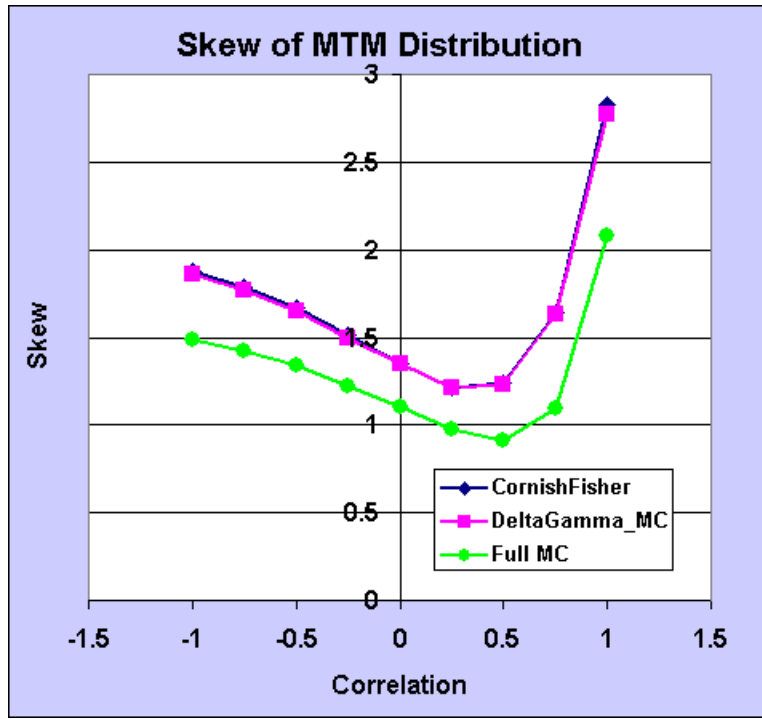


FIGURE 4: Computed Skew for Three Nonlinear VaR Methods. (*MTM* is a traders' synonym for the portfolio value.)

test case, as in the previous one, the Delta-Gamma method exhibits incorrect behavior in the hedged situation. The Delta and Delta-Gamma methods overpredict the VaR for most values of the correlation coefficient. The Cornish-Fisher method slightly overpredicts the VaR and the Delta-Gamma Monte Carlo method slightly underpredicts the VaR. It is interesting to note that except for the hedged or near-hedge situation, the three parametric or matrix methods become increasing accurate as more terms are added to the methodology. Notice that an exact analytic expression (cf. [3]) for the VaR of a portfolio of two calls or puts on perfectly correlated ($\rho = 1$) assets can be obtained. We therefore highlight the corresponding value with a square. The full Monte Carlo simulation VaR value is in excellent agreement with this exact analytic value.

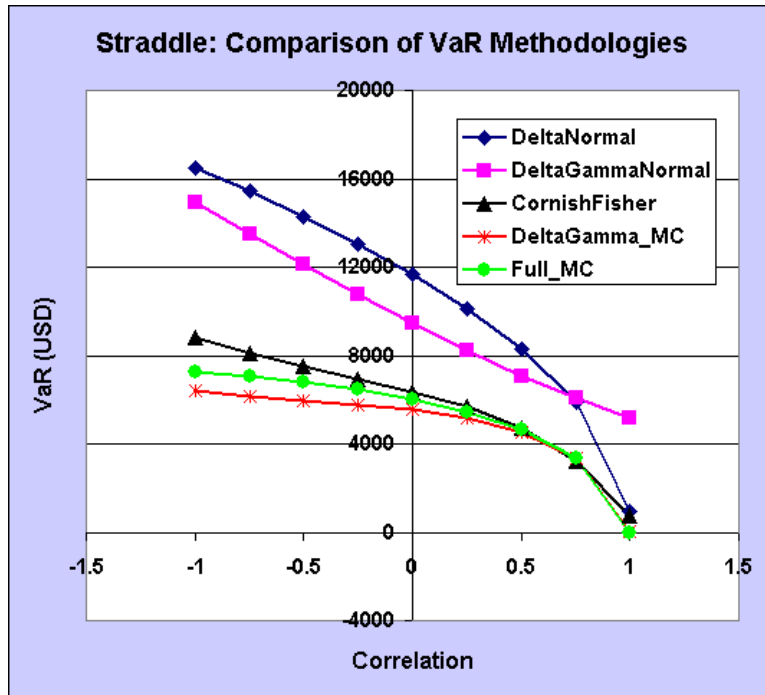


FIGURE 5: Comparison of VaR Methodologies for Test Case 2: ATM Straddle.

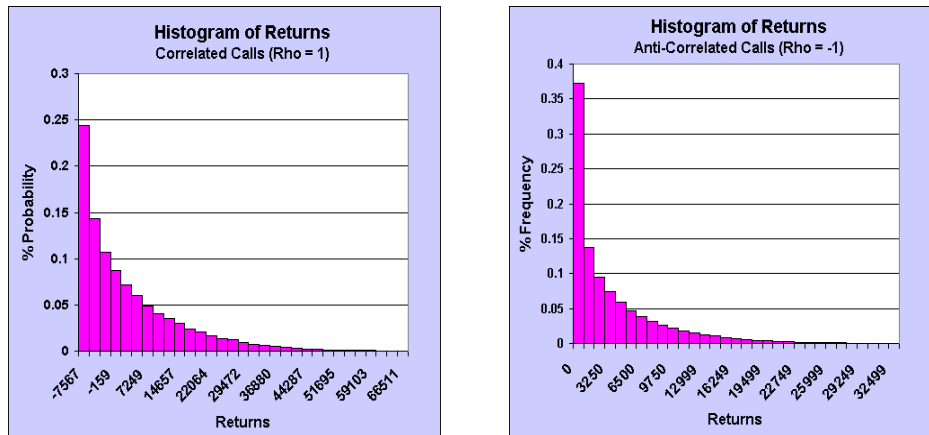


FIGURE 6: Histograms of Returns for Test Case 3: ATM Calls or Puts.

Parameter	Symbol	Value (Unit)
Times to expiry (both legs)	$T_1 = T_2$	0.1 (yr.)
Initial Prices (both legs)	$P_1 = P_2$	100 (\$)
Strike Price	X	0 (\$)
Volatility (1st leg)	$\sigma_{\text{OPT-1}}$	0.3
Volatility (2nd leg)	$\sigma_{\text{OPT-2}}$	0.2
VaR volatility (1st leg)	$\sigma_{\text{VaR-1}}$	0.15
VaR volatility (2nd leg)	$\sigma_{\text{VaR-2}}$	0.1
Notionals (both legs)	$Not_1 = Not_2$	1000

TABLE 3: List of the Parameter Values for the Spread Option Test Case

5.4. Test Case 4: ATM Spread Option. In this case, an ATM Spread option is used to compare the VaR methods. The parameter and input values²² for this case are listed in table 3.

The payoff of this option is proportional to the difference of the price of the two underlying assets. Figure 8 shows two returns histograms for perfect correlation and anti-correlation of the two underlying assets computed using full Monte Carlo simulation.

In this case, unlike the other test cases, the option value depends on two underlying assets and, as a result, can have a non zero cross gamma term. In all of the previous test cases, the option value only depended on one underlying asset and hence, the cross gamma term is zero. Figure 9(a) shows a comparison of the 5 VaR methodologies. The Delta-Gamma, Cornish-Fisher, and Delta-Gamma Monte Carlo methodologies were computed only including the diagonal Gamma terms. For this option, both the Cornish-Fisher and the Delta-Gamma Monte Carlo methods exhibit significant differences in the VaR as compared to the full Monte Carlo simulation. As with all previous test cases, the Delta-Gamma method exhibits a “strange”

²²Homologous parameters with identical values are listed on one line for brevity. Also the definition of the volatilities is the same as in Table 2.

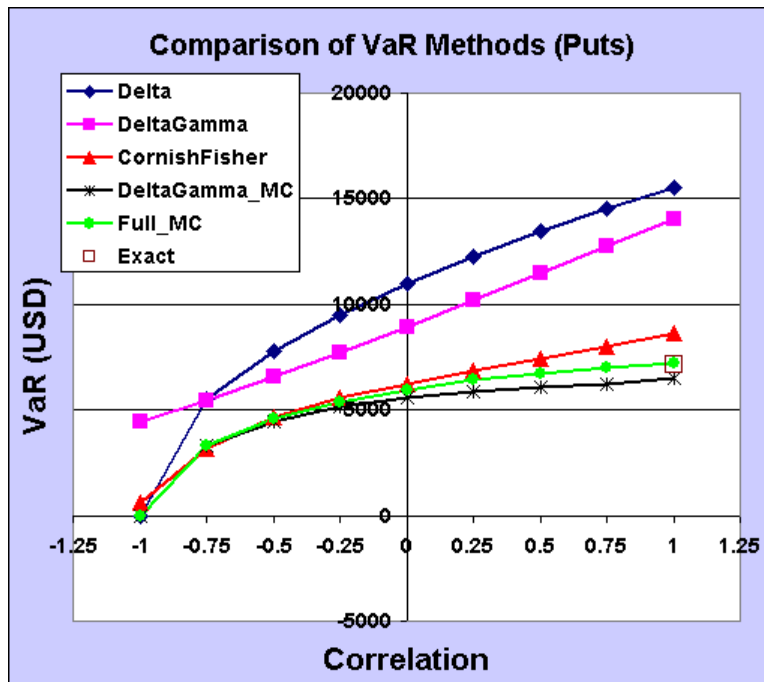
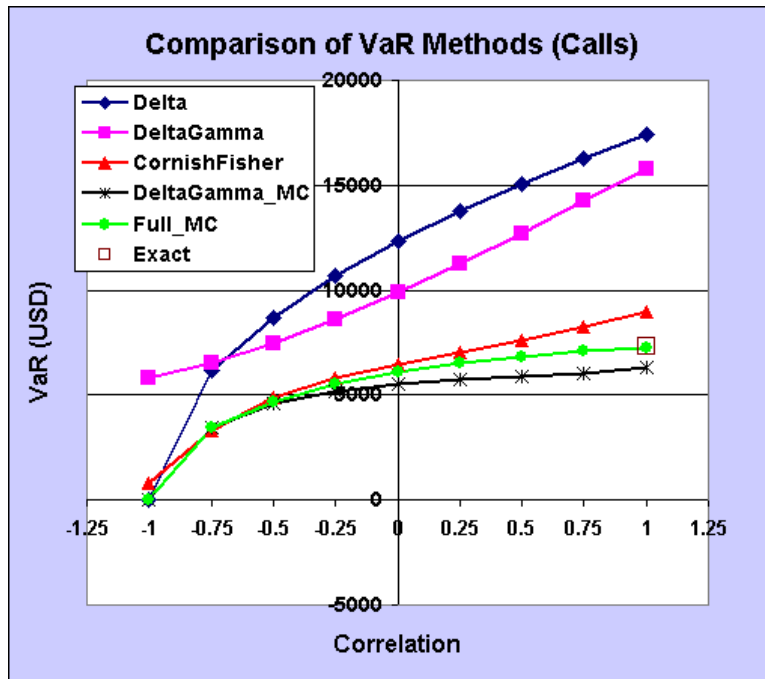


FIGURE 7: Comparison of VaR Methodologies for Test Case 3: Correlated Calls and Puts

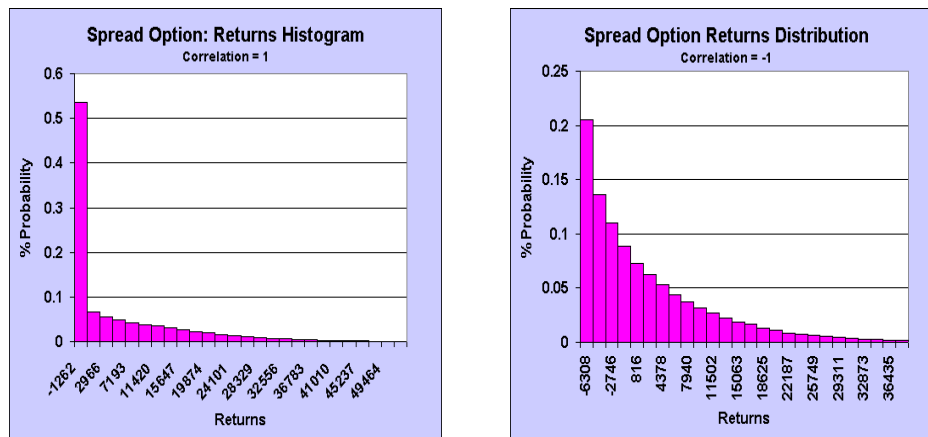
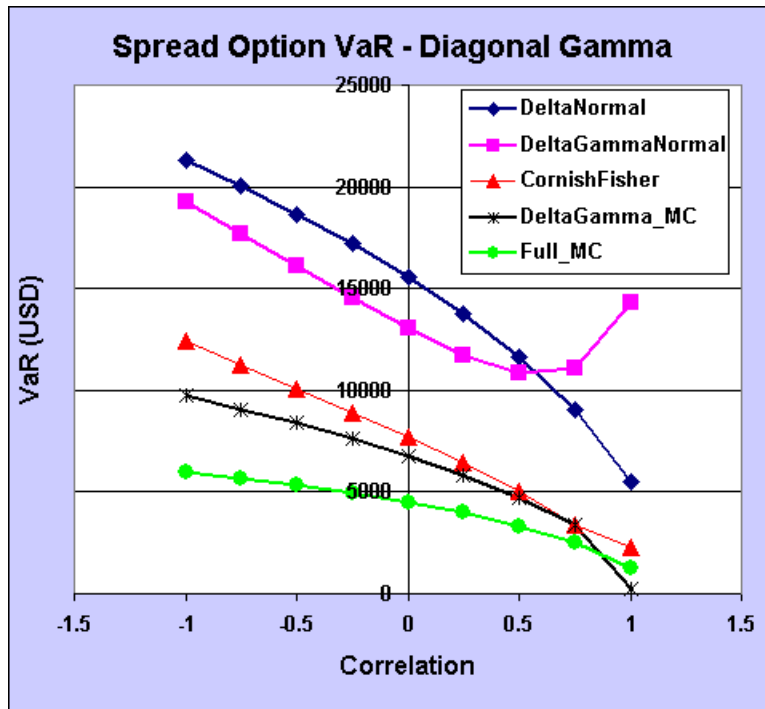


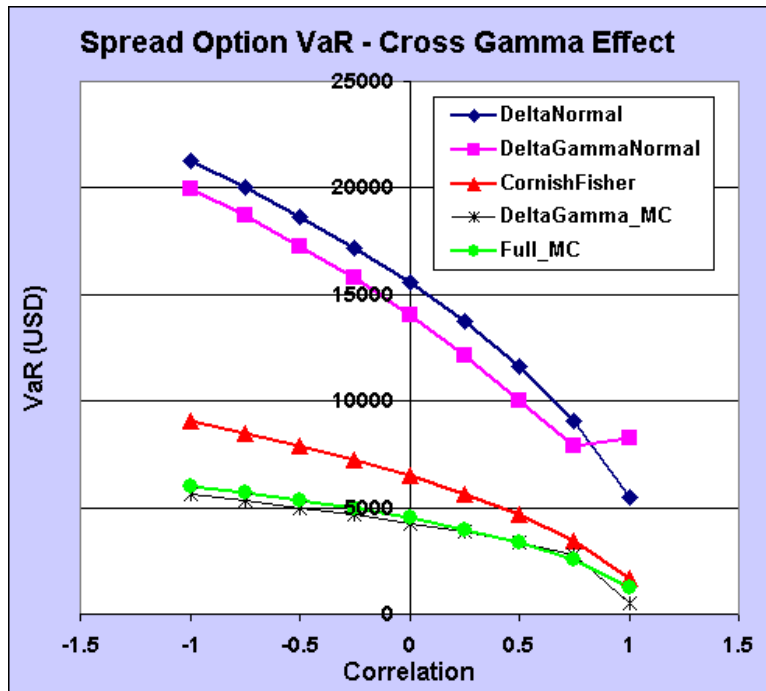
FIGURE 8: Histograms of Returns for Test Case 4: ATM Spread Call Option

and “incorrect” behavior at one of the end points in the correlation space. In this test case, minimal risk or VaR is exhibited when the two underlying assets are fully correlated because of the nature of the ATM spread option payoff.

Figure 9(b) shows a similar comparison of the VaR but in this case the cross gamma terms are included in the Delta-Gamma, Cornish-Fisher, and Delta-Gamma Monte Carlo methods. This seems to have minimal effect on the Delta and Delta-Gamma methods although it improves the Delta-Gamma behavior as the underlying assets become correlated. The accuracy of the Cornish-Fisher is improved but it still overpredicts the VaR. On the other hand, when the cross gamma terms are included, the Delta-Gamma Monte Carlo method is in excellent agreement with the full Monte Carlo simulation although it still slightly underpredicts the VaR.



(A) Neglecting Cross-Gamma Effects



(B) Including Cross-Gamma Effects

FIGURE 9: Comparison of VaR Methodologies for Test Case 4: ATM Spread Call Option.

Parameter	Symbol	Value (Unit)
Times to expiry (all legs)	$T_1 = T_2 = T_3 = T_4$	0.1 (yr.)
Initial Prices (all legs)	$P_1 = P_2 = P_3 = P_4$	100 (\$)
Strike Price (1st leg)	X_1	115 (\$)
Strike Price (2nd leg)	X_2	85 (\$)
Strike Price (3rd and 4th leg)	$X_3 = X_4$	100 (\$)
Volatility (all legs)	$\sigma_{\text{OPT-1}} = \sigma_{\text{OPT-2}} = \sigma_{\text{OPT-3}} = \sigma_{\text{OPT-4}}$	0.3
VaR volatility (all legs)	$\sigma_{\text{VAR-1}} = \sigma_{\text{VAR-2}} = \sigma_{\text{VAR-3}} = \sigma_{\text{VAR-4}}$	0.3
Notionals (1st and 2nd leg)	$Not_1 = Not_2$	-300
Notionals (3rd and 4th leg)	$Not_3 = Not_4$	200

TABLE 4: List of the Parameter Values for Test Case 5

5.5. Test Case 5: A Highly Non-Quadratic Portfolio. In this section we test our Delta-Gamma methodologies with a portfolio which is highly non-quadratic. This consists of two long ATM straddles and three short strangles²³ with strikes symmetric with respect to the straddle's strike. The parameters for this test case are listed in Table 4. The first two legs make up the strangles positions, while the last two comprise the straddles.

The value of the straddle is concave up (or convex) while that of the short positions is concave down (or, in short, concave). The greater number of short positions prevails away from the current price of the underlying thereby making the portfolio value PV concave down there. On the other hand, the straddle values prevail near the money, causing PV to be concave up near the money. Therefore, overall, PV has two negative concavities and one positive. This explains the shape of the graph of PV in Figure 10 (together with the graph of its components). This also makes a quadratic fit of PV as a function on one factor infeasible. The least degree of a good

²³ A strangle comprises a put and call with different strikes and identical expiration and exposure (both long or both short). The payoff at expiration is concave down and piecewise linear. Before expiration, the payoff has the same concavity and it is smooth. Qualitatively it behaves as a quadratic curve with negative leading coefficient. (For some more details on strangles, cf., e.g., [7, Sec. 8.5].)

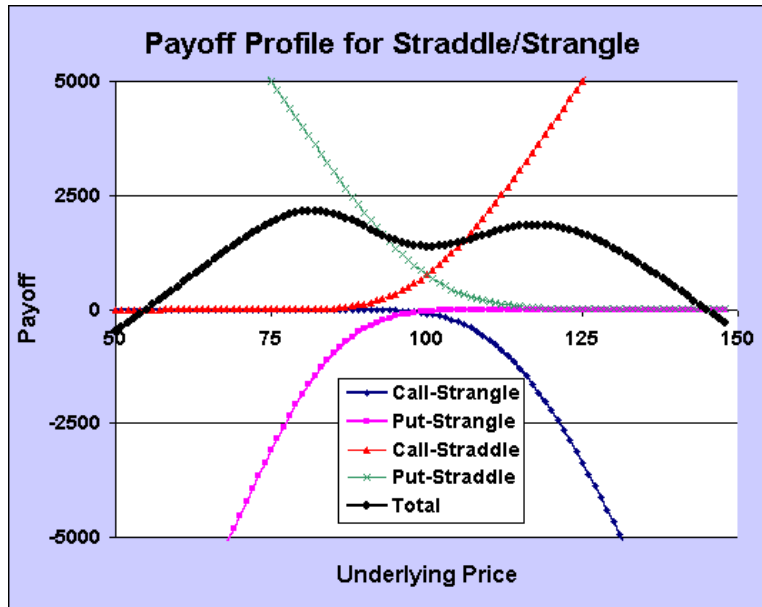


FIGURE 10: Portfolio Values for 2 Long Straddles and Three Short Strangles.

univariate approximation to PV should be four.²⁴ Indeed, for the Delta-Gamma methodologies applied to PV as a univariate function, all of the estimates are inaccurate as compared to full Monte Carlo values.

On the other hand, one notices that such portfolio values are rather uncommon in practice. The reason is that in a large portfolio positions rarely tend to aggregate with such symmetry because the risk factors are different and correlated. At the same time, we note that the same portfolio can be thought of as having the same positions (deals), but depending on as many i.i.d. risk factors. This assumption more faithfully reflects the general situation and

²⁴We synthesized our portfolio value function to replicate the first example of Frye in [6]. Frye exhibited that as a typical example in which Delta-Gamma VaR methodologies would fail. However, we found that increasing dimensions extends the reliability of the approximations that these methodologies give.

Methodology	One-Factor	I.I.D. Multi-Factors
Full Monte Carlo	-3246.16	-14294.78
Delta-Gamma Monte Carlo	-701.25	-13908.79
Delta-Normal	-739.34	-9151.68
Delta-Gamma-Normal	-726.64	-14258.93
Cornish-Fisher	-703.89	-14018.95

TABLE 5: Signed VaR for a highly non-quadratic PV

gives the same full Monte Carlo simulation. It turns out that this modification to the *naive* application of Delta-Gamma methodologies gives also much more accurate VaR estimates. The reason is that by increasing the dimension of our risk factor space, the portfolio is now given by the values of a function $\widetilde{PV} = \widetilde{PV}(X_1, X_2, X_3, X_4)$ over the curve $X_1 = X_2 = X_3 = X_4$. This can be thought of as the diagonal section of a 4-dimensional hypersurface in \mathbb{R}^5 . The quadratic approximation is now a quadratic form in four variables. As such, this form can accurately model different concavities in different dimensions (as many as the gammas in the 4×4 Gamma-matrix $\widetilde{\Gamma}$, which in this case is diagonal). As a result, all Delta-Gamma methodologies have much better accuracies as evidenced in Table 5 (note that we did not distinguish between the diagonal and full Delta-Gamma methods because in this particular case they coincide).

This example illustrates how a wise and realistic choice of risk factors can dramatically improve Delta-Gamma VaR. In particular, these methodology enjoy a *blessing of dimensionality*.²⁵

6. COMPUTATIONAL EFFICIENCY

Each VaR method has its own level of accuracy as well as its own level of computational effort with full Monte Carlo simulation having the greatest accuracy for nonlinear portfolios but also requiring the most computational effort. Modern day portfolios are becoming increasingly large in size—portfolios comprising 25,000 to 100,000 deals are not unusual. In

²⁵We ironically paraphrase a famous limitation of lattice methods.

addition, many of these deals are options and include complicated exotic options like spread options, basket options, average rate Asian options and other path dependent options. Some of these options do not lend themselves to analytical treatment, approximate or otherwise, and require numerical methods such as lattices (e.g. binomial, trinomial, or finite-difference) or Monte Carlo methods. In the future, the VaR method that is used may have to be a compromise between accuracy and computational speed. Obviously, it can't take more than 24 hours to compute tomorrow's VaR.

Computational times are presented only on a relative basis since they are inherently dependent on the specific hardware platform on which the tests are performed.²⁶ Unless otherwise specified, all Monte Carlo simulations are performed on 1,000 trials.

To understand the relative computational effort required by these methods, two portfolios were studied. Portfolio 1 is a synthetic portfolio where there are as many options in the portfolios as risk factors. The number of risk factors and options vary from 100 to 1000. Portfolio 2, on the other hand, contains many more option deals than risk factors. The two portfolios can be summarized as follows:

Portfolio 1:

Number of Risk Factors (N) = Number of Transactions

Example: N = 100 Deals = 100

... ..
N = 1000 Deals = 1000

Portfolio 2:

Number of Transactions \gg Number of Risk Factors

Example: N = 100 Deals = 10,000

... ..
N = 1000 Deals = 10,000

To make it even more interesting, the options are all evaluated using binomial lattices or trees. All the deals involve options depending on one

²⁶The "Moore Law" uptrend in computational speed is also another reason for deeming absolute computational effort time devoid of significance. Suffice it to say that such times have been roughly halved since the first version of this paper was submitted.

underlying asset, hence only the diagonal terms are nonzero. Also the correlation among the individual risk factor is assumed to be zero. However, the algorithms employed in the VaR computation were wholly general and did not take any advantage of the specific form of the covariance and gamma matrices so that the testing of the computational burden is not distorted.²⁷

6.1. Portfolio 1. Figure 11 shows the relative computational times for all five VaR methods scaling the full Monte Carlo simulation as 100% and plotted versus the number of risk factors. In the Delta-Gamma Monte Carlo and full Monte Carlo methods, most of the computational effort is used up in the decomposition of the correlation matrix that is necessary for generating correlated random risk factors in the trials. The exact revaluation of even 1000 deals is minor in comparison as indicated by the Delta-Gamma Monte Carlo and full Monte Carlo times. Cornish-Fisher, in comparison, requires only about 50% of the computational effort and the Delta-Gamma method requiring only about 25% in comparison to the full Monte Carlo simulation.

6.2. Portfolio 2. Figure 12 shows the relative computational times for Portfolio 2 containing 10,000 options valued with binomial methods and for which the Monte Carlo was run using only 50 trials. In this case, the computational time is overwhelmed by the full numerical lattice revaluation carried out for each trial in the full Monte Carlo method. Only 50 trials were used because running the simulation was impractical for 1000 trials in the full Monte Carlo method. To see why, assume that one option can be valued using a binomial lattice method in about 0.01 sec of computational time. The portfolio of 10,000 of these options revalued for 1000 trials requires 10 million numerical evaluations each requiring 0.01 second. This would still require about 27.7 hours of computational time which is impractical. By the same token, this figure support the case either for usage of analytic models

²⁷In truth, the only reason why such matrices were selected is the difficulty in producing synthetic Gamma and covariance matrices of such a size.

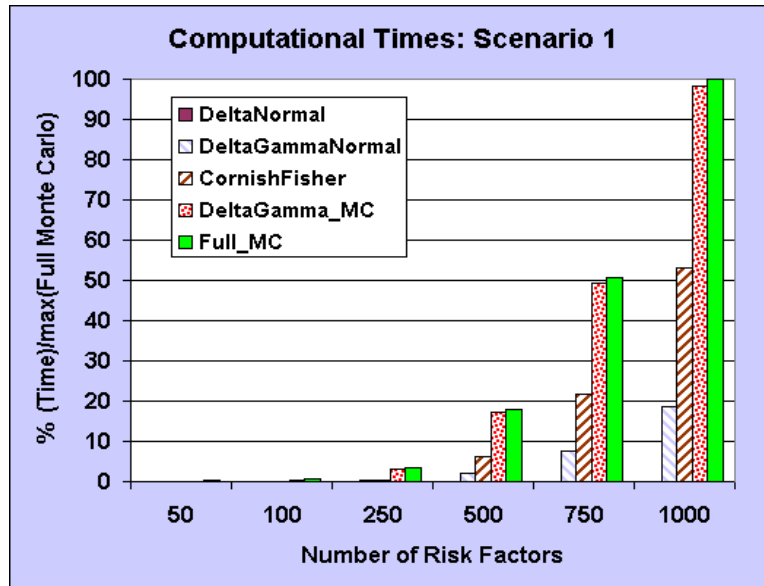


FIGURE 11: Relative Computational Times for Portfolio 1.

within full Monte Carlo simulation or for more computationally efficient alternatives to full Monte Carlo simulation itself.

As a consequence, for this portfolio Monte Carlo Delta-Gamma simulation replaces full Monte Carlo simulation as the comparison benchmark, and, in addition, only the three parametric methods were tested. The relative computational times are illustrated in Figure 13 for 1000 Monte Carlo simulations. In this figure, Delta-Gamma Monte Carlo was scaled as the 100% computational effort. Cornish-Fisher requires about 75% of the computational effort required by the Delta-Gamma Monte Carlo method and the Delta-Gamma method requires about 50%.

7. CONCLUDING REMARKS ON VaR CALCULATIONS

Two observations were evident from this study of five VaR methodologies. The Delta-Gamma-Normal methodology yielded *incorrect* behavior at one of the endpoints of the “correlation space,” and in particular, for a

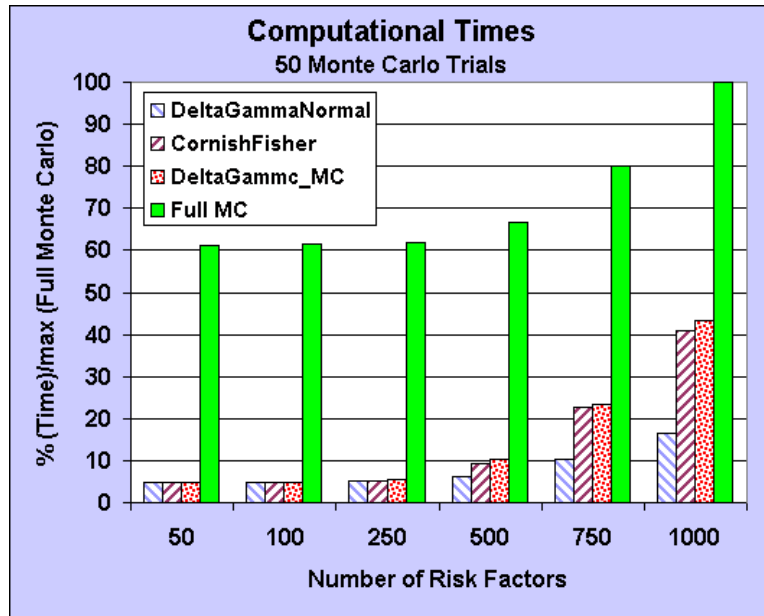


FIGURE 12: Relative Computational Times for Portfolio 2 Using 50 Monte Carlo Simulations.

hedged portfolio situation. The simpler linear Delta method gave better and more meaningful results at this endpoint condition in comparison to the nonlinear Delta-Gamma-Normal method.

The Delta-Gamma Monte Carlo method yields good accuracy with reasonable computational times. Good accuracy is achieved when cross gamma effects are included. In general, the full gamma matrix will be sparse and a *smart implementation* (cf. Section 4) must be employed to achieve low computational effort with high accuracy. In other words, the cross gamma terms can not be blindly computed for all options even if they are negligible since this procedure would require as much time as carrying out a full Monte Carlo simulation.

In general the parametric or matrix methods overpredict the VaR and the Delta-Gamma Monte Carlo method slightly underpredicts the VaR.

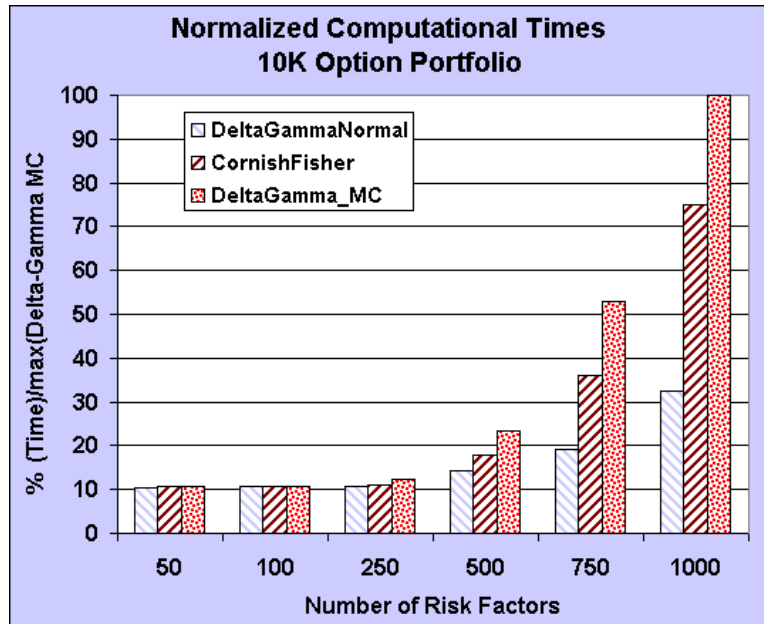


FIGURE 13: Relative Computational Times for Portfolio 2 Using 1000 Delta-Gamma Monte Carlo Simulations.

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E-mail address, Michael J. Siclari: msiclari@olf.com; *Phone*: 516.394.1297